A note about the t'Hooft's ansatz for $SU(N)$ real time guage theories.

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The t'Hooft's ansatz reduces the classical Yang–Mills theory to the $\lambda\phi^4$ one. It is shown that in the frame of this ansatz the real-time classical solutions for the arbitrary $SU(N)$ gauge group is obtained by embedding $SU(2) \times SU(2)$ into $SU(N)$. It is argued that this group structure is the only possibility in the frame of the considered ansatz. New explicit solutions for $SU(3)$ and $SU(5)$ gauge groups are shown.

1 Introduction

In order to simplify the problem of solving a Yang–Mills equation for the vector field it was offered by t'Hooft *at. al.* the ansatz for the Euclidean space [1]. It reduces the Yang–Mills equation to the equation for a single scalar field ϕ. The $SU(2)$ classical solutions discovered by means of this ansatz are well known [2] and were used to generate $SU(N)$ solutions by simply embedding $SU(2)$ into $SU(N)$ [3].

One of them allows the coordinate transformation to the Minkowski space so that it becomes nonsingular, real and possesses a finite action and energy [4, 2].

The $SU(2)$ gauge group was assumed for both the Euclidean and Minkowski space, see also [5], while the experimental analysis shows that QCD is the $SU(3)$ gauge theory [6]. So, the knowledge of the real-time classical solution for QCD is important since it allows to analyze the non-perturbative corrections [7] to the observables.

In this article we will try to find a $SU(N)$ solution by means of the t'Hooft's ansatz. The only condition we assume for the ansatz is the following: it must reduce the Yang–Mills equation to the real scalar $\lambda\phi^4$ theory. We will solve this condition and will show that the only solution of the classical Yang–Mills equation in the frame of the t'Hooft's ansatz is embedding $SU(2) \times SU(2)$ into $SU(N)$.

2 Definition of ansatz

Let us start from the Yang–Mills equation in the matrix form

$$\partial^\mu F_{\mu\nu} + ig[A^\mu, F_{\mu\nu}] = 0, la \tag{1}$$

where

$$A_\mu = t_a A_{a\mu},$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu],$$

t_a are generators of the gauge group.

Let us consider the t'Hooft's ansatz without any assumptions about gauge group

$$A_\mu(x) = \frac{1}{g}\eta_{\mu\nu}\,\partial^\nu \ln\phi(x),$$

where $\eta_{\mu\nu}$ are some matrixes. We will consider that $A_\mu(x)$ satisfies the Lorentz gauge condition: $\partial^\mu A_\mu = 0$ and so $\eta_{\mu\nu}$ are antisymmetric over μ and ν matrixes. It is assumed that $\eta_{\mu\nu}$ are constant in this gauge.

It is necessary to take the equality

$$-i[\eta_{\mu\sigma}, \eta_{\nu\rho}] = \eta_{\mu\nu}g_{\rho\sigma} - \eta_{\mu\rho}g_{\sigma\nu} + \eta_{\sigma\rho}g_{\mu\nu} - \eta_{\sigma\nu}g_{\mu\rho} \tag{2}$$

in order to reduce the Yang–Mills equation to the equation for the single scalar field. As the result of substitution of ansatz with the property (2) into Yang–Mills equation (**??**), we have

$$\Box\phi + \lambda\phi^3 = 0, \tag{3}$$

where λ is an arbitrary integration constant. Emphasize that the Eq.(3) is the result of (2), this reduction is valid for any gauge group.

Therefore, the problem (**??**) was divided into two parts: the searching of $\eta_{\mu\nu}$ from the algebraic equality (2) and the solving of the equation (3) for $\phi(x)$.

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Particular solutions of the equation (3) are known, see [2, 4, 8, 9], and we will not consider this question.

The matrices $\eta_{\mu\nu}$ can be written in a convenient form

$$\eta_{\mu\nu} = -\varepsilon_{0\mu\nu\kappa} X_\kappa + i g_{0\mu} Y_\nu - i g_{0\nu} Y_\mu, \qquad \kappa = 1, 2, 3, \tag{4}$$

since they are antisymmetric, where $\varepsilon_{0123} = 1$; the unknown X_i and Y_i are matrixes in the group space, $X_0 = 0$, $Y_0 = 0$, $X_i = -X^i$, $Y_i = -Y^i$.

Let us insert (4) into (2). Then we obtain algebraic equations for X_i and Y_i. Because of antisymmetry of $\eta_{\mu\nu}$, it is convenient to examine only three cases

1. $\mu = 0, \sigma = i, \nu = 0, \rho = j$, where $i, j = 1, 2, 3$. Then we have

$$[Y_i, Y_j] = i\varepsilon_{ijk} X_k; \tag{5}$$

2. $\mu = 0, \sigma = i, \nu = j, \rho = k$, where $i, j, k = 1, 2, 3$. It is easy to obtain

$$\varepsilon_{jks}[Y_i, X_s] = iY_j g_{ik} - iY_k g_{ij}.$$

So, we have:

$$[Y_i, X_j] = i\varepsilon_{ijk} Y_k; \tag{6}$$

after changing the indexes

3. $\mu = i, \sigma = j, \nu = k, \rho = s$, where $i, j, k, s = 1, 2, 3$. This case gives

$$-i[(-\varepsilon_{ijp} X_p), (-\varepsilon_{ksl} X_l)] = (-\varepsilon_{ikp} X_p) g_{sj} - (-\varepsilon_{isp} X_p) g_{jk} + (-\varepsilon_{jsp} X_p) g_{ik} - (-\varepsilon_{jkp} X_p) g_{is}$$

After symplification and changing indexes we have

$$[X_i, X_j] = i\varepsilon_{ijk} X_k. \tag{7}$$

The other cases can be easily reduced to this three ones.

It follows from (5,6,7) that

$$[\mathcal{J}_i, \mathcal{J}_j] = i\varepsilon_{ijk}\mathcal{J}_k \qquad [\mathcal{K}_i, \mathcal{K}_j] = i\varepsilon_{ijk}\mathcal{K}_k, lb \tag{8}$$

$$[\mathcal{J}_i, \mathcal{K}_j] = 0,$$

where

$$\mathcal{J}_i = \frac{X_i + Y_i}{2}, \qquad \mathcal{K}_i = \frac{X_i - Y_i}{2}.$$

It follows from (**??**) that $N \times N$ matrixes $\mathcal{J}_i$ and $\mathcal{K}_i$ are elements of the $SU(2) \times SU(2)$ group. Then the ansatz can be written as follows

$$\eta_{\mu\nu} = (-\varepsilon_{0\mu\nu\kappa}\mathcal{J}_\kappa + i g_{0\mu}\mathcal{J}_\nu - i g_{0\nu}\mathcal{J}_\mu) + (-\varepsilon_{0\mu\nu\kappa}\mathcal{K}_\kappa - i g_{0\mu}\mathcal{K}_\nu + i g_{0\nu}\mathcal{K}_\mu), \qquad \kappa = 1, 2, 3. \tag{9}$$

This is the general solution of (2) and, therefore, it is unique. There always exists a nonzero t'Hooft's ansatz for any $N \geq 2$ since the representation of the $SU(2) \times SU(2)$ group by $N \times N$ matrixes always exists. The meaning of such representation is embedding $SU(2) \times SU(2)$ into $SU(N)$.

This ansatz gives complex potentials A_μ for real ϕ, however one can check that it leads to a real Lagrangian density. Therefore one can expect that there exists some complex gauge transformation which makes it real as it was done for $SU(2)$ [4].

Let us consider the solutions for $SU(2)$, $SU(3)$ and $SU(5)$ groups.

2.1 SU(2)

For the $SU(2)$ gauge group the only solution is (either $\mathcal{J}_i$ or $\mathcal{K}_i$ is equal to zero)

$$X_i = \pm Y_i = \frac{\sigma_i.}{2}$$

Then we obtain well-known solution [1, 2] which can be written in a component form:

$$\eta_{a\mu\nu} = -\varepsilon_{0a\mu\nu} \mp i g_{0\mu} g_{a\nu} \pm i g_{0\nu} g_{a\mu}.$$

2.2 SU(3)

For the $SU(3)$ gauge group also either $\mathcal{J}_i$ or $\mathcal{K}_i$ is equal to zero, so we have

$$X_i = \pm Y_i.$$

There exists both reducible and irreducible representation of the $SU(2)$ group in terms of 3×3 matrixes.

2.2.1 Reducible representation

The $SU(3)$ group contains 3 independent $SU(2)$ subgroups which are not form direct product. So there exist 3 independent solutions:
(I): $X_1^{(I)} = t_1, X_2^{(I)} = t_2, X_3^{(I)} = t_3$
In the component form we obtain

$$\eta_{1\,\mu\nu} = \begin{pmatrix} 0 & \pm i & 0 & 0 \\ \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{2\,\mu\nu} = \begin{pmatrix} 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & 1 \\ \mp i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{3\,\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \pm i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \mp i & 0 & 0 & 0 \end{pmatrix}_{\mu\nu},$$

$$\eta_{a\,\mu\nu} = 0, a = 4, ..., 8.$$

(II): $X_1^{(II)} = t_4, X_2^{(II)} = t_5, X_3^{(II)} = \frac{1}{2}(\sqrt{3}\,t_8 + t_3)$;
(III): $X_1^{(III)} = t_6, X_2^{(III)} = t_7, X_3^{(III)} = \frac{1}{2}(\sqrt{3}\,t_8 - t_3)$.
The cases (II) and (III) are similar to the (I) with the difference in gauge indexes.

2.2.2 Irreducible representation

There also exist irreducible representation of the $SU(2)$ group by 3×3 matrixes.

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then in the component form we obtain

$$\eta_{1\,\mu\nu} = \sqrt{2} \begin{pmatrix} 0 & \pm i & 0 & 0 \\ \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{2\,\mu\nu} = \sqrt{2} \begin{pmatrix} 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & 1 \\ \mp i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{3\,\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \pm i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \mp i & 0 & 0 & 0 \end{pmatrix}_{\mu\nu},$$

$$\eta_{4\,\mu\nu} = \eta_{5\,\mu\nu} = 0, \quad \eta_{6\,\mu\nu} = \eta_{1\,\mu\nu}, \quad \eta_{7\,\mu\nu} = \eta_{2\,\mu\nu}, \quad \eta_{8\,\mu\nu} = \sqrt{3}\,\eta_{3\,\mu\nu}.$$

2.3 SU(5)

Considering the $SU(5)$ group it is interesting to examine the solution with both nonzero $SU(2)$ groups. If $\mathcal{J}_i$ or $\mathcal{K}_i$ is equal to zero then the solution will be given by reducible or irreducible representation of the group in a way like $SU(3)$.

For the $\mathcal{J}_i$ one can take irreducible group presentation for the 3×3 matrixes in the upper left corner and for the $\mathcal{K}_i$ one can take 2×2 group presentation for the lower right corner, and vice versa. It can be written in the obvious form:

$$\begin{pmatrix} \mathcal{J} & 0 \quad 0 \\ SU(2) & 0 \quad 0 \\ 3 \times 3 & 0 \quad 0 \\ 0 \quad 0 \quad 0 & \mathcal{K} \\ 0 \quad 0 \quad 0 & SU(2) \quad 2 \times 2 \end{pmatrix}$$

Then the ansatz in component form is as follows:

$$\eta_{1\,\mu\nu} = \sqrt{2} \begin{pmatrix} 0 & \pm i & 0 & 0 \\ \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{2\,\mu\nu} = \sqrt{2} \begin{pmatrix} 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & 1 \\ \mp i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{3\,\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \pm i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \mp i & 0 & 0 & 0 \end{pmatrix}_{\mu\nu},$$

$$\eta_{4\,\mu\nu}=\eta_{5\,\mu\nu}=0,\quad \eta_{6\,\mu\nu}=\eta_{1\,\mu\nu},\quad \eta_{7\,\mu\nu}=\eta_{2\,\mu\nu},\quad \eta_{8\,\mu\nu}=\sqrt{3}\,\eta_{3\,\mu\nu},\quad \eta_{9\ldots 20\,\mu\nu}=0,$$

$$\eta_{21\,\mu\nu}=\begin{pmatrix}0&\mp i&0&0\\ \pm i&0&0&0\\ 0&0&0&-1\\ 0&0&1&0\end{pmatrix}_{\mu\nu},\quad \eta_{22\,\mu\nu}=\begin{pmatrix}0&0&\mp i&0\\ 0&0&0&1\\ \pm i&0&0&0\\ 0&-1&0&0\end{pmatrix}_{\mu\nu},\quad \eta_{23\,\mu\nu}=\begin{pmatrix}0&0&0&\mp i\\ 0&0&-1&0\\ 0&1&0&0\\ \pm i&0&0&0\end{pmatrix}_{\mu\nu},$$

$$\eta_{24\,\mu\nu}=0.$$

If one believes that the $SU(5)$ theory is unification of electro–weak and strong interactions then indexes $a=1,...,8$ correspond to the strong and $a=21,...,23$ to the electro–weak interactions. But one can see that this solution can not be used for this purpose.

3 Conclusions

In the frame of the ansatz the $SU(N)$ classical solutions always exist and each one is given by embedding $SU(2)\times SU(2)$ into $SU(N)$.

Let us assume that ϕ is invariant under $O(4)\times O(2)$ coordinate transformations [4, 9]. In the frame of this prescription, we obtain the real solution of the Yang-Mills equation

$$A_0=\pm\frac{x_0x_a}{g\,y^2}\mathcal{J}_a\mp\frac{x_0x_a}{g\,y^2}\mathcal{K}_a,$$

$$A_i=\frac{1}{g\,y^2}\left[-\varepsilon_{ain}x_n\pm\delta_{ai}\frac{1}{2}(1+x^2)\pm x_ax_i\right]\mathcal{J}_a+\frac{1}{g\,y^2}\left[-\varepsilon_{ain}x_n\mp\delta_{ai}\frac{1}{2}(1+x^2)\mp x_ax_i\right]\mathcal{K}_a,$$

where

$$y^2=\frac{1}{4}(1-x^2)^2+x_0^2,\qquad \varepsilon_{123}=1,\quad n=1...3,$$

and $\mathcal{J}_a$, $\mathcal{K}_a$ are corresponding representation of $SU(2)\times SU(2)$ group by $N\times N$ matrixes.

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